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Readers' Forum

Brief discussion of previous investigations in the aerospace sciences and technical comments on papers published in the AIAA Journal *are presented in this special department. Entries must be restricted to a maximum of 1000 words, or the equivalent of one* Journal *page including formulas and figures. A discussion will be published as quickly as possible after receipt of the manuscript. Neither the AIAA nor its editors are responsible for the opinions expressed by the correspondents. Authors will be invited to reply promptly.*

Comment on "Vibration and Buckling of Flexible Rotating Beams"

Alexander H. Flax*
Potomac, Maryland 20854

EICK and Mignolet[1] have investigated singular and regular perturbation methods as applied to the natural vibrations of rotating beams. However, accurate numerical methods, as efficient as contemporary computation technology would admit, have been available for many decades. For example, the 1944 book by Myklestad[2] provided practical methods of numerical analysis for vibration of rotating propellers, turbine blades, and wings, including the effects of coupling of bending and torsional deflections of these components. (The computation technology of that era was exemplified by the Marchant and Friden desk calculating machines.)

Thus, analyses of approximate analytical solutions such as those provided by perturbation methods are mainly of theoretical interest but may sometimes contribute insights into the limitations of numerical methods or means of improving their accuracy or efficiency. Approximate analytical solutions also on occasion may lead to simplified formulas and methods adequate for preliminary design, for delineating ranges of parameters in which approximations are valid, or for providing a rough check on more elaborate calculations. The aim herein is to show how Southwell's theorem on the eigenvalue of the first mode of a vibrating system whose structural stiffness can be represented as a sum of constituent parts—and its empirical extension to higher vibration modes—can complement perturbation methods and also help to provide limits for their ranges of accuracy.

Southwell's theorem[3] applied to the first eigenvalue (frequency squared) of a rotating beam, λ_1, gives a lower bound for λ_1, as

$$\lambda_1 \geq \lambda_{c1} + \lambda_{b1} = \lambda_{s1} \cdots \tag{1}$$

where λ_{c1} is the lowest eigenvalue of a rotating chain or cable of zero bending stiffness having the same mass distribution as the beam, and λ_{b1} is the lowest eigenvalue of the nonrotating beam. Defining the bending stiffness parameter as in Ref. 1,

$$\varepsilon = EI/m\Omega^2 R^2$$

where EI is the beam bending stiffness, m mass per unit length, Ω the angular velocity of rotation, and R the tip radius of the rotating member. For a uniform beam, fixed at the center of rotation and free at the tip, Southwell's theorem leads to

$$\lambda_1' = \omega^2/\Omega^2 \geq 1 + 12.36\varepsilon = \lambda_{s1}' \cdots \tag{2}$$

where λ_1' is the eigenvalue λ_1 nondimensionalized with respect to the square of the angular velocity of rotation.

It is also well known empirically, as a result of numerical comparisons over a wide range of cases, that the Southwell sum of constituent eigenvalues for the nth mode of vibration λ_{sn} is a good approximation to the nth eigenvalue of the rotating beam when $\lambda_{bn} > \lambda_{cn}$, so that

$$\lambda_n' \approx \lambda_{sn}' = \lambda_{cn}' + \lambda_{bn}' \cdots \tag{3}$$

up to high values of the mode number n. (For reference, a cable or chain of uniform mass[4] has successive values of λ_c of 1, 6, 15, and 28. A cantilever beam of uniform mass and stiffness[5] has successive values of λ_b, of 12.37, 485.48, 3807.02, and 14619.72.) An approximate value for the second eigenvalue of a rotating cantilever beam of uniform mass and stiffness is then

$$\lambda_2' \approx \lambda_{s2}' = 6 + 485.48\varepsilon \cdots \tag{4}$$

A breakpoint in the accuracy of the approximation of the foregoing equations occurs when the eigenvalue of the stationary beam in bending becomes less than the corresponding eigenvalue of the rotating cable of zero bending stiffness. For the first vibration mode, from Eq. (2), this occurs at $\varepsilon_1^* = 0.081$. For the second mode, from Eq. (4), this occurs at $\varepsilon_2^* = 0.0124$.

From the results given by Eick and Mignolet,[1] it can be observed that this breakpoint criterion applies in reverse to the results of singular perturbation calculation; i.e., the range of validity of the singular perturbation approximation is for $\varepsilon_n < \varepsilon_n^*$. Values of λ_1' and λ_2' from the exact and singular perturbation results of Ref. 1, along with values based on Southwell sum calculations in Tables 1 and 2, illustrate these points. For values of ε above ε^* (below the dotted line), values of λ_n' based on Southwell sums are always more accurate than those derived from singular perturbation theory, which become less accurate as ε increases.

Table 1 Mode 1 eigenvalues for $a = 0$ ($\varepsilon^* = 0.081$)

ε	Exact	Singular perturbation	Southwell sum
0.001	1.07	1.07	1.01
0.004	1.15	1.15	1.05
0.010	1.25	1.24	1.12
0.040	1.66	1.54	1.49
....			
0.100	2.42	1.95	2.24
0.400	6.14	3.48	5.94
1.000	13.55	5.96	13.36
4.000	50.64	16.61	49.94

Table 2 Mode 2 eigenvalues for $a = 0$ ($\varepsilon^* = 0.012$)

ε	Exact	Singular perturbation	Southwell sum
0.001	6.78	6.89	6.48
0.004	8.36	8.86	7.94
0.010	11.3	12.5	10.9
....			
0.040	25.9	29.8	25.4
0.100	55.0	63.5	55.0
0.400	200.7	229.1	200.2
1.000	492.0	557.2	491.5
4.000	1948.6	2188.6	1947.9

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*Consultant, 9007 Belmart Road. Fellow AIAA.

Table 3 Mode 1 eigenvalues for $a = 0.5$ ($\varepsilon^* = 0.0126$)

ε	Exact	Singular perturbation	Southwell sum
0.001	2.86	2.79	2.70
0.004	3.51	3.14	3.29
0.010	4.72	3.60	4.48
....			
0.040	10.67	5.08	10.41
0.100	22.54	7.24	22.28
0.400	81.88	15.56	81.60

Table 4 Mode 1 eigenvalues for $a = 0.9$ ($\varepsilon^* = 0.00012$)

ε	Exact	Singular perturbation	Southwell sum
....			
0.001	139	38	138
0.004	510	79	507
0.010	1,252	149	1,250
0.040	4,960	460	4,958
0.100	12,378	1,042	12,375
0.400	49,465	3,830	49,454

Since λ'_{bn} increases more rapidly than λ'_{cn} as n increases, it is apparent that ε_n^* becomes monotonically smaller as the mode number n increases. Hence, in terms of ε, the range of validity of singular perturbation theory decreases with mode number. In contrast to the criterion for validity of the singular perturbation methods employed by Eick and Mignolet, the ε_n^* criterion does not fail for the first mode when the beam is fixed at the axis of rotation. However, the magnitudes of ε_n^* are generally consistent with the ε ranges for validity of the singular perturbation method discussed by the authors.

For the cases in which the beam is fixed outboard of the center of rotation at distance a, expressed as a fraction of the tip radius R, the Southwell sum for the first mode for uniform mass and bending stiffness can be expressed approximately as

$$\lambda'_{s1} = \left[1 + \tfrac{3}{2}a/(1-a)\right] + 12.36\varepsilon/(1-a)^4 \cdots \tag{5}$$

The first term is an approximate expression for λ_c, the lowest frequency of a rotating cable attached at point a. It is an approximation to the exact eigenvalue $\lambda'_c = \nu(\nu+1)/2$ for $P\nu(a) = 0$ where $P(\nu)$ is the Legendre function or order ν (and where ν need not be an integer). Comparisons of the results from the first term of Eq. (5) with exact values indicate an error of less than 2% for $a < 0.6$ and about 3% at $a = 0.9$. These errors are on the high side since the approximation for λ_{c1} is based on a rigid rod, hinged at point a, and represents, in effect, a Rayleigh quotient for the lowest cable eigenvalue. With this approximation, the Southwell sum for the lowest eigenvalue is no longer a rigorous lower bound. The second term adjusts the beam bending frequency formula for $a = 0$ to account for the reduced beam length from the fixed point at a.

Tables 3 and 4 further demonstrate the significance of ε^* as a breakpoint for the accuracy of the Southwell sum approximation and also, in an inverse way, for the accuracy of the singular perturbation approximation, for $a \neq 0$. It is interesting that in the case of $a = 0.90$ in Table 4, ε^* is so much less than the values of ε tabulated that Southwell sums give accurate results for the lowest eigenvalues within 1% of exact values throughout whereas the singular perturbation results are wildly inaccurate for all values of ε in the table.

References

[1]Eick, C. D., and Mignolet, M. P., "Vibration and Buckling of Flexible Rotating Beams," *AIAA Journal*, Vol. 33, No. 3, 1995, pp. 528–538.

[2]Myklestad, N. O., *Vibration Analysis*, McGraw–Hill, New York, 1994, pp. 210–214, 239, 240.

[3]Den Hartog, J. P., *Mechanical Vibrations*, 2nd ed., McGraw–Hill, New York, 1948, pp. 307–310.

[4]Morse, P. M., *Vibration and Sound*, 2nd ed., McGraw–Hill, New York, 1948, pp. 114–118.

[5]Timoshenko, S., *Vibration Problems in Engineering*, 2nd ed., Van Nostrand, New York, 1937, pp. 114–118.

Reply by the Authors to A. Flax

Chris D. Eick*
AlliedSignal Engines, Phoenix, Arizona 85072-2181
and
Marc P. Mignolet†
Arizona State University, Tempe, Arizona 85287-6106

THE comment by Flax[1] provides an interesting combined use of the singular perturbation method and of Southwell's approximation but also indirectly reemphasizes two important conclusions that were drawn in our original paper,[2] namely the high accuracy and usefulness of the regular perturbation approach and the importance of the parameter εn^2 where $\varepsilon = EI/m\Omega^2\bar{R}^4$ (see Ref. 2) and n is the mode number. These issues are discussed further below.

Accuracy and Usefulness of the Regular Perturbation Approach

It was found in Ref. 2 that the natural frequencies and mode shapes of a rotating beam can be predicted accurately for essentially all values of the normalized beam stiffness ε and clamping radius a by relying on the regular perturbation method described therein. In particular, it was shown that the eigenvalues (squares of the natural frequencies) admit a series representation of the form

$$\lambda = \left(\lambda_{-1}/\varepsilon^{\frac{1}{3}}\right) + \lambda_0 + \lambda_1\varepsilon^{\frac{1}{3}} + \cdots \tag{1}$$

where the coefficients λ_i, $i = -1, 0, 1, \ldots$ are functions of the sole parameter

$$\Delta = (1-a)/\varepsilon^{\frac{1}{3}} \tag{2}$$

Further, it was shown that the eigenvalue estimates computed on the basis of only the first two terms in Eq. (1) are extremely reliable for all clamping radii a and normalized beam stiffnesses ε investigated. This surprising result then motivated the determination of the asymptotic behavior of the coefficients λ_{-1} and λ_0 as $\Delta \to 0$ and $\Delta \to \infty$, which can be expressed as[2]

$$\lim_{\Delta\to 0} \lambda_{-1} = \left(\beta_n^4/\Delta^4\right) + (\bar{\gamma}_n/\Delta) \tag{3}$$

$$\lim_{\Delta\to 0} \lambda_0 = \delta_n \tag{4}$$

where β_n are the standard nonrotating clamped-free beam frequencies and

$$\bar{\gamma}_n = 2 - \alpha_n\beta_n + \frac{\alpha_n^2\beta_n^2}{2} \tag{5}$$

$$\delta_n = -\tfrac{3}{4} + (\alpha_n\beta_n/2)[1 - (\alpha_n\beta_n/3)] \tag{6}$$

with

$$\alpha_n = \frac{\cos(\beta_n) + \cosh(\beta_n)}{\sin(\beta_n) + \sinh(\beta_n)} \tag{7}$$

and

$$\lim_{\Delta\to\infty} \lambda_{-1} = \mu_n^2/4\Delta \tag{8}$$

$$\lim_{\Delta\to\infty} \lambda_0 = -\tfrac{1}{6}\left[1 + \left(\mu_n^2/4\right)\right] \tag{9}$$

where μ_n designates the nth zero of the Bessel function J_0.

It was observed in our original investigation[2] that the limits given by Eqs. (3–7) were valid in the range $\Delta \in [0, 3]$ while Eqs. (8)

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*Senior Engineer, Component Design. Associate Member AIAA.

†Associate Professor, Department of Mechanical and Aerospace Engineering.